Monotonicity of Permanents of Certain Doubly Stochastic Circulant Matrices

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ABSTRACT

Let $p_k(A)$, $k=2,\dots,n$, denote the sum of the permanents of all $k\times k$ submatrices of the $n\times n$ matrix A. A conjecture of Đoković, which is stronger than the famed van der Waerden permanent conjecture, asserts that the functions $p_k((1-\theta)J_n+\theta A)$, $k=2,\dots,n$, are strictly increasing in the interval $0\leqslant\theta\leqslant 1$ for every doubly stochastic matrix A. Here J_n is the $n\times n$ matrix all whose entries are equal $1/n$. In the present paper it is proved that the conjecture holds true for the circulant matrices $A=\alpha I_n+\beta P_n$, $\alpha,\beta\geqslant 0$, $\alpha+\beta=1$, and $A=(nJ_n-I_n-P_n)/(n-2)$, where I_n and P_n are respectively the $n\times n$ identity matrix and the $n\times n$ permutation matrix with 1's in positions $(1,2),(2,3),\dots,(n-1,n),(n,1)$.

1. INTRODUCTION

Let Ω_n be the set of all $n\times n$ doubly stochastic matrices, let J_n be the $n\times n$ matrix all whose entries are equal to $1/n$, let I_n be the $n\times n$ identity matrix, and let P_n be the $n\times n$ permutation matrix with 1's in positions $(1,2),(2,3),\dots,(n-1,n),(n,1)$.

Let A be an $n\times n$ matrix. Denote by $p_k(A)$, $k=1,\dots,n$, the sum of the permanents of all $\binom{n}{k}^2$ $k\times k$ submatrices of A, and define $p_0(A)=1$. Note that $p_n(A)=p(A)$ is the permanent of A.

The famed van der Waerden conjecture asserts that if $A\in\Omega_n$, then

$$p(A)\geqslant p(J_n)=\frac{n!}{n^n},$$

with equality if and only if $A=J_n$.

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A stronger version of this conjecture [1] states that the functions

$$p_k((1-\theta)J_n+\theta A), \qquad k=2,\dots,n,$$

where A is any fixed matrix on the boundary of Ω_n, are strictly increasing in the interval $0\leqslant\theta\leqslant 1$. In [2] the case $k=n$ of the above assertion was proved for $A=I_n$ and $A=nJ_n-I_n$. In [5, p. 158, Problem 8] the problem of finding other matrices A for which this assertion holds was posed, and in [3] it was proved to hold for $A=(I_n+P_n)/2$.

In the present paper we prove this assertion for $A=\alpha I_n+\beta P_n$ where $\alpha,\beta\geqslant 0$, $\alpha+\beta=1$, and for $A=(nJ_n-I_n-P_n)/(n-2)$.

The subsequent discussion relies strongly on [3]. We state now Lemmas 1 and 5 of [3] as our Lemmas 1 and 2 respectively and mention some other results needed later.

LEMMA 1. *Let $n\geqslant 3$, and let $A\in\Omega_n$. If*

$$\frac{p_{k-1}(A)}{p_{k-1}(J_n)}\leqslant\frac{p_k(A)}{p_k(J_n)}, \qquad k=2,\dots,n, \tag{1}$$

with strict inequality for $2\leqslant k\leqslant n-1$, then the functions

$$p_k((1-\theta)J_n+\theta A), \qquad k=2,\dots,n,$$

are strictly increasing in the interval $0\leqslant\theta\leqslant 1$.

LEMMA 2. *Let $n\geqslant 2$. Then*

$$p_k(I_n+P_n)=\begin{cases}\dfrac{n}{n-k}\dbinom{2n-k-1}{k}, & k=0,\dots,n-1,\\ 2, & k=n.\end{cases} \tag{2}$$

Đoković [1] conjectured that (1) holds for all $A\in\Omega_n$ and proved his conjecture for $k=2$ and $k=3$.

Note that

$$p_k(J_n)=\binom{n}{k}^2\frac{k!}{n^k}. \tag{3}$$

Let $A \in \Omega_n$, and denote

$$h_{A,k}(\theta) = p_k((1-\theta)J_n + \theta A), \qquad k = 2, \dots, n.$$

By [4, Lemma 2],

$$h_{A,k}(\theta) = p_k(J_n) \sum_{i=0}^{k} \binom{k}{i} (1-\theta)^{k-i} \theta^i \frac{p_i(A)}{p_i(J_n)},$$

and so

$$h'_{A,k}(\theta) = k p_k(J_n) \sum_{i=1}^{k-1} \binom{k-1}{i} (1-\theta)^{k-i-1} \theta^i \left[\frac{p_{i+1}(A)}{p_{i+1}(J_n)} - \frac{p_i(A)}{p_i(J_n)} \right]. \qquad (4)$$

In Sec. 2 we consider the circulant doubly stochastic matrices $A = \alpha I_n + \beta P_n$, $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$, and obtain formulas for $p_k(\alpha I_n + \beta P_n)$, $k = 0, \dots, n$ (Lemma 3). In the proof of these formulas we refer to a combinatorial result included in the proof of Lemma 5 of [3] (but not stated explicitly there). Using these formulas, we prove that the functions

$$p_k((1-\theta)J_n + \theta(\alpha I_n + \beta P_n)), \qquad k = 2, \dots, n,$$

are strictly increasing in the interval $0 \leqslant \theta \leqslant 1$ (Theorem 1).

In Sec. 3 we consider the matrices $A = (nJ_n - I_n - P_n)/(n-2)$, $n \geqslant 3$, and prove that the functions

$$p_k\Big((1-\theta)J_n + \frac{\theta}{n-2}(nJ_n - I_n - P_n)\Big), \qquad k = 2, \dots, n,$$

are strictly increasing in the interval $0 \leqslant \theta \leqslant 1$ (Theorem 2). The proof is based on properties of a family of polynomials $g_{k,n}(x)$ (Lemmas 3–5). Note that the permanent of $nJ_n - I_n - P_n$ is related to the "problème des ménages."

2. $A = \alpha I_n + \beta P_n$

In this section we consider the $n \times n$ doubly stochastic circulant matrices $\alpha I_n + \beta P_n$. We first obtain formulas for

$$p_k(\alpha I_n + \beta P_n), \qquad k = 0, \dots, n.$$

LEMMA 3. *Let $n \geqslant 2$, and let α, β be given numbers. Then*

$$p_k(\alpha I_n + \beta P_n) = \begin{cases} \dfrac{n}{n-k} \displaystyle\sum_{l=0}^{k} \binom{n-l-1}{n-k-1} \binom{n-k+l-1}{n-k-1} \alpha^l \beta^{k-l}, & k=0,\dots,n-1, \\ \alpha^n + \beta^n, & k=n. \end{cases} \tag{5}$$

Proof. Equation (5) is easily verified for $k=0$ and $k=n$.

Let $1 \leqslant k \leqslant n-1$. It is obvious that

$$p_k(\alpha I_n + \beta P_n) = \sum_{l=0}^{k} N_l(n,k) \alpha^l \beta^{k-l}, \tag{6}$$

where $N_l(n,k)$ is the number of different diagonals of 1's of length k in $I_n + P_n$ which occupy precisely l positions of I_n (and so $k-l$ positions of P_n). Diagonals of length k in the $n \times n$ matrix $I_n + P_n$ are defined in the obvious way.

$N_l(n,k)$ was implicitly found in the proof of Lemma 5 in [3]. Using the notation and results there, we get

$$N_l(n,k) = \sum_{m=0}^{\min(l,n-k)} \binom{n}{l,m} \binom{n-l-m}{k-l}$$

$$= \frac{n}{n-k} \binom{n-l-1}{n-k-1} \binom{n-k+l-1}{n-k-1}. \tag{7}$$

Equation (5) follows from (6) and (7). ∎

THEOREM 1. *Let $n \geqslant 3$, let α, β be nonnegative numbers, $\alpha+\beta=1$, and let $A = \alpha I_n + \beta P_n$. Then the functions*

$$h_{A,k}(\theta) = p_k((1-\theta)J_n + \theta A), \qquad k=2,\dots,n,$$

are strictly increasing in the interval $0 \leqslant \theta \leqslant 1$.

Proof. By Lemma 1, it is sufficient to show that

$$\frac{p_{k-1}(\alpha I_n+\beta P_n)}{p_{k-1}(J_n)}\leqslant\frac{p_k(\alpha I_n+\beta P_n)}{p_k(J_n)},\qquad k=2,\dots,n, \tag{8}$$

with strict inequality for $2\leqslant k\leqslant n-1$.

We first prove (8) for $4\leqslant k\leqslant n-2$. So let

$$4\leqslant k\leqslant n-2. \tag{9}$$

Using (5) and $\alpha+\beta=1$, we obtain

$$\begin{aligned}
p_{k-1}(\alpha I_n+\beta P_n)&=\frac{n(\alpha+\beta)}{n-k+1}\sum_{l=0}^{k-1}\binom{n-l-1}{n-k}\binom{n-k+l}{n-k}\alpha^l\beta^{k-l-1}\\
&=\frac{n}{n-k+1}\left[\sum_{l=0}^{k-1}\binom{n-l-1}{n-k}\binom{n-k+l}{n-k}\alpha^l\beta^{k-l}\right.\\
&\qquad\left.+\sum_{l=0}^{k-1}\binom{n-l-1}{n-k}\binom{n-k+l}{n-k}\alpha^{l+1}\beta^{k-l-1}\right]\\
&=\frac{n}{n-k+1}\sum_{l=0}^{k}\left[\binom{n-l-1}{n-k}\binom{n-k+l}{n-k}+\binom{n-l}{n-k}\binom{n-k+l-1}{n-k}\right]\alpha^l\beta^{k-l}.
\end{aligned} \tag{10}$$

Here $\binom{n-k-1}{n-k}$ is taken as 0. From (3), (5), and (10) follows

$$\begin{aligned}
&\frac{p_k(\alpha I_n+\beta P_n)}{p_k(J_n)}-\frac{p_{k-1}(\alpha I_n+\beta P_n)}{p_{k-1}(J_n)}\\
&=c_{k,n}\sum_{l=0}^{k}\left\{\frac{n}{n-k}\binom{n-l-1}{n-k-1}\binom{n-k+l-1}{n-k-1}\right.\\
&\qquad\left.-\frac{n-k+1}{k}\left[\binom{n-l-1}{n-k}\binom{n-k+l}{n-k}+\binom{n-l}{n-k}\binom{n-k+l-1}{n-k}\right]\right\}\alpha^l\beta^{k-l}\\
&=\frac{c_{k,n}}{k}\sum_{l=0}^{k}\binom{n-l}{n-k}\binom{n-k+l}{n-k}\\
&\qquad\times\frac{kn(n-k)-(n-k+1)[(n-k+l)(k-l)+(n-l)l]}{(n-l)(n-k+l)}\alpha^l\beta^{k-l},
\end{aligned} \tag{11}$$

where

$$c_{k,n}=\frac{k!n^k[(n-k)!]^2}{(n!)^2}.$$

As

$$(n-k+l)(k-l)+(n-l)l,$$

regarded as a function of l, attains its maximum at $l=k/2$, it follows that

$$(n-k+l)(k-l)+(n-l)l\leqslant\frac{k(2n-k)}{2}. \tag{12}$$

Hence, using (9) and (12),

$$\begin{aligned}
&kn(n-k)-(n-k+1)[(n-k+l)(k-l)+(n-l)l]\\
&\quad\geqslant\frac{k}{2}[2n(n-k)-(n-k+1)(2n-k)]=\frac{k}{2}[n(k-2)+k-k^2]\\
&\quad\geqslant\frac{k}{2}[(k+2)(k-2)+k-k^2]=\frac{k(k-4)}{2}\geqslant 0.
\end{aligned}$$

So

$$kn(n-k)-(n-k+1)[(n-k+l)(k-l)+(n-l)l]\geqslant 0, \tag{13}$$

with equality if and only if $k=4$, $l=2$, and $n=6$.

From (11) and (13) follows (8) with strict inequality, and so (8) is proved for $4\leqslant k\leqslant n-2$.

For $k=2$ and $k=3$ the proof of (8) is straightforward, and we omit it. Note also that for $k=2$ and $k=3$ the Đoković conjecture (1) was proved [1] for any doubly stochastic matrix A.

To complete the proof, we have to show that (8) holds for $k=n-1$ and $k=n$ with strict inequality for $k=n-1$.

For $k=n-1$, (11) becomes

$$\frac{p_{n-1}(\alpha I_n+\beta P_n)}{p_{n-1}(J_n)}-\frac{p_{n-2}(\alpha I_n+\beta P_n)}{p_{n-2}(J_n)}$$
$$=\frac{c_{n-1,n}}{n-1}\sum_{l=0}^{n-1}\left[(n-1)(n-2)-4l(n-1)+4l^2\right]\alpha^l\beta^{n-1-l}.$$

Substituting $\alpha/\beta=x$, we get

$$\frac{p_{n-1}(\alpha I_n+\beta P_n)}{p_{n-1}(J_n)}-\frac{p_{n-2}(\alpha I_n+\beta P_n)}{p_{n-2}(J_n)}=\frac{c_{n-1,n}\beta^{n-1}}{n-1}f(x),$$

where

$$f(x)=(n-1)(n-2)\sum_{l=0}^{n-1}x^l-4(n-1)\sum_{l=0}^{n-1}lx^l+4\sum_{l=0}^{n-1}l^2x^l. \tag{14}$$

To prove (8) for $k=n-1$, we have to show that $f(x)>0$ for $0\leqslant x<\infty$. Set

$$g(x)=\sum_{l=0}^{n-1}x^l=\frac{x^n-1}{x-1},\qquad x\neq 1. \tag{15}$$

We have

$$f(x)=(n-1)(n-2)g(x)-4(n-1)xg'(x)+4x[xg'(x)]'. \tag{16}$$

A straightforward computation, using (15) and (16), gives

$$h(x)=f(x)(x-1)^3$$
$$=(n-1)(n-2)x^{n+2}-2(n-2)(n+1)x^{n+1}+(n^2+n+2)x^n$$
$$-(n-1)(n-2)+2(n-2)(n+1)x-(n^2+n+2)x^2.$$

As $h'''(x)>0$ for $x>0$ and as $h(1)=h'(1)=h''(1)=0$, it follows that $h(x)<0$ for $0\leqslant x<1$ and $h(x)>0$ for $x>1$. Hence, $f(x)>0$ for $x>0$, $x\neq 1$. But (14) implies that $f(1)>0$, and so $f(x)>0$ for $x>0$, and the proof of (8) for $k=n-1$ is completed.

For $k=n$, we get from (3) and (5), using $\alpha+\beta=1$,

$$\frac{P_n(\alpha I_n+\beta P_n)}{p_n(J_n)}-\frac{p_{n-1}(\alpha I_n+\beta P_n)}{p_{n-1}(J_n)}$$

$$=\frac{n^{n-1}}{n!}\left[n(\alpha^n+\beta^n)-\sum_{l=0}^{n-1}\alpha^l\beta^{n-1-l}\right]$$

$$=\frac{n^{n-1}}{n!(\beta-\alpha)}\left[n(\alpha^n+\beta^n)(\beta-\alpha)+(\alpha^n-\beta^n)\right]$$

$$=\frac{n^{n-1}}{n!(\beta-\alpha)}\left[n(\alpha^n+\beta^n)(\beta-\alpha)+(\alpha^n-\beta^n)(\alpha+\beta)\right].$$

Substituting $\alpha/\beta=x$, we get

$$\frac{p_n(\alpha I_n+\beta P_n)}{p_n(J_n)}-\frac{p_{n-1}(\alpha I_n+\beta P_n)}{p_{n-1}(J_n)}=\frac{n^{n-1}\beta^{n+1}}{n!(\beta-\alpha)}u(x),\qquad \alpha\neq\beta,$$

where

$$u(x)=(1-n)x^{n+1}+(n+1)x^n-(n+1)x+(n-1).$$

As $u(1)=u'(1)=0$, $u''(x)>0$ for $0<x<1$, and $u''(x)<0$ for $x>1$, it follows that $u(x)>0$ for $0\leqslant x<1$ (i.e., $\alpha<\beta$), and $u(x)<0$ for $x>1$ (i.e., $\alpha>\beta$), and so (8) holds for $k=n$ (with equality if and only if $\alpha=\beta$). The proof of our theorem is thus completed. ■

Note that the theorem holds also for $n=2$ and $\alpha\neq\beta$. For $n=2$ and $\alpha=\beta=\frac{1}{2}$, we have $A=(I_2+P_2)/2=J_2$ and $h_{A,2}(\theta)$ is constant.

The case $\alpha=\beta=\frac{1}{2}$ of Theorem 1 was proved before in [3].

3. $A=(nJ_n-I_n-P_n)/(n-2)$

In this section we consider the $n\times n$ doubly stochastic matrices $(nJ_n-I_n-P_n)/(n-2)$, $n\geqslant 3$.

First we consider a related family of polynomials $g_{k,n}(x)$. In the following lemma two recurrence relations for $g_{k,n}(x)$ are obtained.

LEMMA 4. *Let* $n \geqslant 3$, *and let*

$$g_{k,n}(x) = \sum_{i=1}^{k-1} (-1)^{i+1} \binom{k-1}{i} \frac{i(2n-i-2)!\left[(n-i-1)!\right]^2 (n-i-1)}{(2n-2i-1)!} x^{i-1},$$

$$k = 2, \ldots, n. \quad (17)$$

Then

$$x g'_{k,n}(x) - (k-2) g_{k,n}(x) = -(k-1) g_{k-1,n}(x), \qquad k = 3, \ldots, n, \quad (18)$$

and

$$(k-1) x g'_{k-1,n}(x) - 2(k-1)(n-1) g_{k-1,n}(x) = g'_{k,n+1}(x), \qquad k = 3, \ldots, n+1. \quad (19)$$

Proof. Define

$$f_{k,n}(x) = \sum_{i=1}^{k-1} (-1)^{i+1} \binom{k-1}{i} \frac{(2n-i-2)!\left[(n-i-1)!\right]^2 (n-i-1)}{(2n-2i-1)!} x^{i}. \quad (20)$$

We have

$$f'_{k,n}(x) = g_{k,n}(x).$$

Proof of (18). Noting that

$$\binom{k-1}{i} = \binom{k-2}{i} + \frac{i}{k-1} \binom{k-1}{i},$$

it follows from (17) and (20) that

$$f_{k,n}(x) = f_{k-1,n}(x) + \frac{x}{k-1} g_{k,n}(x). \quad (21)$$

Differentiating (21), (18) follows.

Proof of (19). By (17),

$$g_{k,n+1}(x)=\sum_{i=1}^{k-1}(-1)^{i+1}\binom{k-1}{i}\frac{i(2n-i)![(n-i)!]^2(n-i)}{(2n-2i+1)!}x^{i-1}$$

$$=\sum_{i=0}^{k-2}(-1)^{i+2}\binom{k-1}{i+1}\frac{(i+1)(2n-i-1)![(n-i-1)!]^2(n-i-1)}{(2n-2i-1)!}x^{i}$$

$$=-(k-1)\sum_{i=0}^{k-2}(-1)^{i+1}\binom{k-2}{i}$$

$$\times\frac{(2n-1-i)(2n-i-2)![(n-i-1)!]^2(n-i-1)}{(2n-2i-1)!}x^{i}$$

$$=(k-1)x\sum_{i=1}^{k-2}(-1)^{i+1}\binom{k-2}{i}$$

$$\times\frac{i(2n-i-2)![(n-i-1)!]^2(n-i-1)}{(2n-2i-1)!}x^{i-1}$$

$$-(k-1)(2n-1)\sum_{i=1}^{k-2}(-1)^{i+1}\binom{k-2}{i}$$

$$\times\frac{(2n-i-2)![(n-i-1)!]^2(n-i-1)}{(2n-2i-1)!}x^{i}$$

$$+(k-1)[(n-1)!]^2(n-1).$$

Hence,

$$g_{k,n+1}(x)=(k-1)xg_{k-1,n}(x)-(k-1)(2n-1)f_{k-1,n}(x)$$

$$+(k-1)[(n-1)!]^2(n-1). \tag{22}$$

Differentiating (22), (19) follows. ∎

LEMMA 5. *Let $n \geqslant 3$, and let*

$$g_n = g_{n,n}(1) = \sum_{i=1}^{n-1} (-1)^{i+1} \binom{n-1}{i} \frac{i(2n-i-2)!\,[(n-i-1)!]^2(n-i-1)}{(2n-2i-1)!}.$$

Then

$$g_n \geqslant 0, \qquad n = 3, 4, \ldots, \tag{23}$$

with equality if and only if $n = 4$.

Proof. Denote

$$a_i = (-1)^{i+1} \binom{n-1}{i} \frac{i(2n-i-2)!\,[(n-i-1)!]^2(n-i-1)}{(2n-2i-1)!}, \qquad i = 1, \ldots, n-1.$$

Then

$$\left| \frac{a_{i+1}}{a_i} \right| = \frac{n-i-2}{n-i-1}\,\frac{2n-2i-1}{2n-i-2}\,\frac{2}{i}, \qquad i = 1, \ldots, n-2,$$

and it follows that $|a_1| = |a_2|$ for $n = 4$, $|a_1| < |a_2|$ for $n > 4$, and

$$|a_i| > |a_{i+1}|, \qquad i = 2, \ldots, n-2. \tag{24}$$

As $g_3 = 2$, $g_4 = 0$, $g_5 = 96$, and $g_6 = 1920$, (23) holds for $3 \leqslant n \leqslant 6$. For $n \geqslant 7$, it follows from (24) that

$$g_n \geqslant \sum_{i=1}^{6} a_i.$$

But

$$\sum_{i=1}^{6} a_i = (n-1)!(n-2)!(n-2)$$

$$\times \left[1 - \frac{2(n-3)}{n-2} + \frac{(n-4)(2n-5)}{(n-2)^2} - \frac{2}{3}\,\frac{(n-5)(2n-7)}{(n-2)^2} \right.$$

$$\left. + \frac{1}{3}\,\frac{(n-6)(2n-7)(2n-9)}{(n-2)^2(2n-6)} - \frac{2}{15}\,\frac{(n-7)(2n-9)(2n-11)}{(n-2)^2(2n-6)} \right]$$

$$= \frac{(n-1)!(n-2)!(n-2)(2n^3+16n^2-213n+516)}{15(n-2)^2(2n-6)}.$$

As

$$2n^3+16n^2-213n+516>0$$

for $n\geqslant 7$, (23) holds for $n\geqslant 7$ and the proof of Lemma 5 completed. ■

Using Lemmas 3 and 4, we prove now that $g_{k,n}(x)>0$ for $0<x<1$.

LEMMA 6. *Let $n\geqslant 3$, and let*

$$g_{k,n}(x)=\sum_{i=1}^{k-1}(-1)^{i+1}\binom{k-1}{i}\frac{i(2n-i-2)![(n-i-1)!]^2(n-i-1)}{(2n-2i-1)!}x^{i-1},$$

$$k=2,\dots,n. \quad (17)$$

Then

$$g_{k,n}(x)>0, \qquad 0<x<1. \quad (25)$$

Proof. Using induction on n, we prove that

$$g_{k,n}(x)>0 \text{ and } g'_{k,n}(x)\leqslant 0, \qquad k=2,\dots,n, \quad 0<x<1, \quad (26)$$

with strict inequality for $n>3$ and $3\leqslant k\leqslant n$.

As $g_{2,3}(x)\equiv 1$ and $g_{3,3}(x)\equiv 2$, (26) holds for $n=3$.

Assume that (26) holds for a given n, $n\geqslant 3$, and all k, $k=2,\dots,n$. We prove that (26) holds for $n+1$ and all k, $k=2,\dots,n+1$. From our induction hypothesis and the recurrence relation (19), it follows that $g'_{k,n+1}(x)<0$, $k=3,\dots,n+1$, $0<x<1$. But $g'_{2,n+1}(x)\equiv 0$, and so $g'_{k,n+1}(x)\leqslant 0$, $k=2,\dots,n+1$, $0<x<1$, with strict inequality for $k\geqslant 3$. Using now the recurrence relation (18), we obtain

$$(k-2)g_{k,n+1}(x)-(k-1)g_{k-1,n+1}(x)=xg'_{k,n+1}(x)<0,$$

$$k=3,\dots,n+1, 0<x<1.$$

Hence,

$$(k-2)g_{k,n+1}(x)<(k-1)g_{k-1,n+1}(x), \qquad k=3,\dots,n+1, \quad 0<x<1. \quad (27)$$

From (27) it follows that

$$\frac{k-1}{n} g_{n+1,n+1}(x) < g_{k,n+1}(x), \qquad k=2,\dots,n. \tag{28}$$

As $g'_{n+1,n+1}(x)<0$, $0<x<1$, we have

$$g_{n+1} = g_{n+1,n+1}(1) < g_{n+1,n+1}(x), \qquad 0<x<1. \tag{29}$$

By (23),

$$g_{n+1} \geqslant 0. \tag{30}$$

From (28), (29), and (30) it follows that

$$g_{k,n+1}(x)>0, \qquad k=2,\dots,n+1, \quad 0<x<1.$$

The validity of (26) for $n+1$ is now established, and the proof of Lemma 6 is completed. ■

Note that for $n \neq 4$, (25) holds for $0 \leqslant x \leqslant 1$.

THEOREM 2. *Let $n \geqslant 3$, and let $A=(nJ_n - I_n - P_n)/(n-2)$. Then the functions*

$$h_{A,k}(\theta) = p_k((1-\theta)J_n + \theta A), \qquad k=2,\dots,n,$$

are strictly increasing in the interval $0 \leqslant \theta \leqslant 1$.

Proof. Consider the $n \times n$ matrix $B=(I_n+P_n)/2$. By (2) and (3),

$$\frac{p_{i+1}(B)}{p_{i+1}(J_n)} = \frac{(i+1)![(n-i-1)!]^2 n^{i+2}}{2^{i+1}(n-i-1)(n!)^2}\binom{2n-i-2}{i+1}, \qquad i=1,\dots,n-2.$$

Hence,

$$\frac{p_{i+1}(B)}{p_{i+1}(J_n)} - \frac{p_i(B)}{p_i(J_n)} = \frac{2i(2n-i-2)![(n-i-1)!]^2(n-i-1)}{(n!)^2(2n-2i-1)!}\left(\frac{n}{2}\right)^{i+1},$$
$$i=1,\dots,n-2. \tag{31}$$

From (2) and (3) it follows that

$$\frac{p_n(B)}{p_n(J_n)} - \frac{p_{n-1}(B)}{p_{n-1}(J_n)} = 0,$$

and so (31) holds also for $i = n-1$. Consider

$$h_{B,k}(\theta) = p_k\left[(1-\theta)J_n + \theta\frac{I_n + P_n}{2}\right], \qquad k = 2,\dots,n.$$

By (4),

$$h'_{B,k}(\theta) = kp_k(J_n)\sum_{i=1}^{k-1}\binom{k-1}{i}(1-\theta)^{k-i-1}\theta^i\left[\frac{p_{i+1}(B)}{p_{i+1}(J_n)} - \frac{p_i(B)}{p_i(J_n)}\right],$$

and so, using (31), we obtain

$$h'_{B,K}(\theta) = \frac{knp_k(J_n)(1-\theta)^{k-1}}{(n!)^2}$$

$$\times \sum_{i=1}^{k-1}\binom{k-1}{i}\frac{i(2n-i-2)![(n-i-1)!]^2(n-i-1)}{(2n-2i-1)!}\left[\frac{\theta n}{2(1-\theta)}\right]^i,$$

$$k = 2,\dots,n.$$

Noting that for $\theta = -2/(n-2)$ we have $(1-\theta)J_n + \theta B = (nJ_n - I_n - P_n)/(n-2) = A$, it follows that $h_{A,k}(\theta)$ is strictly increasing in the interval $0 \leqslant \theta \leqslant 1$ if $h'_{B,k}(\theta) < 0$ for $-2/(n-2) < \theta < 0$. Substituting $-\theta n/[2(1-\theta)] = x$, it follows that $h_{A,k}(\theta)$ is strictly increasing in $0 \leqslant \theta \leqslant 1$ if $h'_{B,k}(\theta(x)) < 0$ for $0 < x < 1$. But

$$h'_{B,k}(\theta(x)) = -\frac{kn^kp_k(J_n)x}{(n!)^2(n-2x)^{k-1}}g_{k,n}(x), \qquad k = 2,\dots,n,$$

where $g_{k,n}(x)$ are the polynomials defined by (17). Using now (25), it follows that $h'_{B,k}(\theta(x)) < 0$ for $0 < x < 1$, and the proof of Theorem 2 is completed. ∎

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Received 31 March 1980; revised 16 June 1980